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# Free algebras with an empty set of generators

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MATHEMATICS

# FREE ALGEBRAS WITH AN EMPTY SET OF GENERATORS

BY

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The purpose of this note is to unify some well-known concepts of algebra, such as direct product, tensor product and embedding, with the use of the concept of a free algebra. This is done by extending the set of operations and axioms of the given algebraic structure in a suitable way with new constants and axioms. The above-mentioned concepts are then obtained as free algebras with 0 generators for the new algebraic structure. General theorems about existence and uniqueness of free algebras (cf. [2]) then yield existence and uniqueness of direct products and tensor products. Some basic properties are obtained immediately by translating basic properties of free algebras. In the case of embedding the results are less satisfactory; here we have to know the possibility of embedding to be able to get the universal embedding, which can be mapped homomorphically into every embedding. Moreover the well-known classical example of the quotient field is not covered by our theorem, as neither the axioms of a domain of integrity nor the axioms of a field satisfy our conditions.

Uniqueness of concepts as mentioned above is discussed in a somewhat more general situation in [1], appendice III. We shall direct our attention more to the problem of existence, although uniqueness is also obtained in a way, which is different from the method of [1]. The present note contradicts Bourbaki's remark, that such an existence proof is only possible if the structure is given explicitly.

For terminology and notation we refer to our paper [2]. For our present purposes, however, it is necessary to consider the occurrence of empty sets somewhat more carefully. In the first place we allow the order of an operation to be zero. A zero-ary operation is called a *constant*. Also empty sets of operations and empty algebras are allowed. An empty $V$-algebra is possible if and only if $V$ contains no constants. The set $W$ of variables used for the definition of $V$–$W$-polynomials also may be empty. If $W$ is empty, there are no $V$–$W$-polynomials of order 0; in that case the set of $V$–$W$-polynomials is empty if and only if $V$ contains no constants. The constant symbols are polynomials of order 1. The generating set of a $V$-algebra may be empty; in that case the algebra is empty if and only if $V$ contains no constants. An axiom system may be empty. Finally a free $Q$–$V$-algebra with 0 generators may be considered. Only in the case that $V$ contains constants this concept is of interest.

Concerning consistency we adopt a terminology which differs from the one used in [2]. We now call an axiom system $Q$ *inconsistent* if there are no $Q$–$V$-algebras and otherwise *consistent*. If a $Q$–$V$-algebra exists with at least two elements, we call $Q$ *strongly consistent*. The last concept coincides with what was called consistent in [2].

In the proofs of lemma 1 and theorem 3 of [2] the strong consistency was only used to guarantee that the variables of $W_n$ would be inequivalent and therefore that $B_n$ would have $n$ generators and not less. If $n=0$ or 1, this is not necessary and so for these numbers of generators theorem 3 holds already, if $Q$ is consistent instead of strongly consistent (in the new sense).

The following condition for an axiom system $Q$ is referred to in the following with (A).

(A) The $V$–$W$-axiom system $Q$ is consistent and contains only axioms of the form $L_1 \vee \ldots \vee L_m$ with $m \geqslant 1$, in which all $L_i$ are identities or negations of identities; in every axiom at most one of the $L_i$ is an identity.

We use the following theorem which is a direct consequence of theorem 3 of [2] and of the above remark.

Theorem 1. If a $V$–$W$-axiom system $Q$ satisfies (A), a free $Q$–$V$-algebra with 0 generators exists.

For convenience we shall often index a set $V$ of operations in the well-known manner by choosing an index set $I$ of the same cardinal number as $V$ and a one-to-one mapping $i \to O_i$ of $I$ onto $V$. So $V$ becomes a family of operations $O_i$ with index set $I$.

We now start with a $V$–$W$-axiom system $Q$, which satisfies (A). We choose a new set $W_1$ of variables containing $W$ and a new set $V_1$ of operations containing $V$, with the restriction that all operations of $V_1$ which are not in $V$ are constants. Finally we take a $V_1$–$W_1$-axiom system $Q_1$. We index the complement of $V$ in $V_1$ with an index set $I_1$. A $V_1$-algebra $B$ now may be interpreted in the obvious way as a $V$-algebra together with a mapping $\varphi$ of $I_1$ into $B$; we say that $\varphi$ is associated to $B$. If the $V_1$-algebra $B$ is a $Q_1$–$V_1$-algebra, we say that the associated mapping is an *admissible* mapping; obviously in that case $B$ satisfies $Q$. In the examples discussed below a simple description of admissible mappings will be possible. A $V_1$-algebra is generated by the empty set if and only if it is generated as a $V$-algebra by the image of its associated mapping. A mapping $\vartheta$ of a $V$-algebra $B$ with associated mapping $\varphi$ into a $V$-algebra $C$ with associated mapping $\psi$ is a $V_1$-homomorphism if and only if it is a $V$-homomorphism and $\vartheta\varphi=\psi$.

The following lemma is a consequence of theorem 1.

Lemma 1. If $Q_1$ satisfies (A), a $Q$–$V$-algebra $F$ with an admissible associated mapping $\varphi$ exists, which is generated by the image of $\varphi$, with the property that for every $Q$–$V$-algebra $B$ with an admissible associated mapping $\psi$ a $V$-homomorphism $\vartheta$ of $F$ into $B$ exists, which satisfies $\vartheta\varphi = \psi$.

The algebra $F$ is uniquely determined up to isomorphism by theorem 1 of [2].

We discuss some applications of lemma 1. We start with the concept of a free product.

Let $Q$ be a $V$–$W$-axiom system, which satisfies (A) and with the property that the one-element $V$-algebra is a $Q$–$V$-algebra (in order that these conditions are satisfied it is sufficient that all axioms of $Q$ are identities; such an algebraic structure is usually called equational). Let $V$ be indexed with an index set $I$. Let there be given a family $A_j (j \in J)$ of disjoint $Q$–$V$-algebras. The operations of $A_j$ are denoted by $O_{ji} (i \in I)$ and are $k_i$-ary, the corresponding operation symbols are denoted by $\Omega_i$. For every $j \in J$ and every $a \in A_j$ we take a constant $a'$, and obtain $V_1$ by adding these constants to $V$. The set of these constants can be indexed by the union $I_1$ of all $A_j$. For every $i \in I$, $j \in J$ and $a_\nu \in A_j (\nu = 1, \ldots, k_i)$ we put

$$\Omega_i (a_1', \ldots, a_{k_i}') = \left( O_{ji} (a_1, \ldots, a_{k_i}) \right)'$$

and obtain $Q_1$ by adding these identities as axioms to $Q$. Obviously the one-element $V_1$-algebra satisfies $Q_1$ and so $Q_1$ satisfies (A). A mapping $\varphi$ of $I_1$ into a $V$-algebra is an admissible mapping if and only if for every $j \in J$ the restriction $\varphi_j$ to $A_j$ of $\varphi$ is a $V$-homomorphism. The following theorem is a consequence of lemma 1.

Theorem 2 (*on free products*). If $Q$ is a $V$–$W$-axiom system, which satisfies (A), if the one-element $V$-algebra is a $Q$–$V$-algebra, if $A_j$ $(j \in J)$ is a family of $Q$–$V$-algebras, a $Q$–$V$-algebra $F$ and a family $\varphi_j$ $(j \in J)$ of $V$-homomorphisms of $A_j$ into $F$ exist with the property that $F$ is generated by the union of the images of $\varphi_j$, and that for every $Q$–$V$-algebra $B$ with a family $\psi_j$ $(j \in J)$ of $V$-homomorphisms of $A_j$ into $B$, a $V$-homomorphism $\vartheta$ of $F$ into $B$ exists satisfying $\vartheta\varphi_j = \psi_j$ for $j \in J$.

The algebra $F$ of theorem 2 is called the *free product* of the family $A_j$.

The most interesting case is that the $\varphi_j$ are isomorphisms. We prove that a sufficient condition is that every $A_j$ contains a one-element subalgebra $\{o_j\}$. For a fixed $r \in J$ we take $B = A_r$ and the family $\psi_j$, defined by $\psi_j(a) = o_r$ for $j \in J$, $j \neq r$, $a \in A_j$, and $\psi_r(a) = a$ for $a \in A_r$. Let $\chi_r$ be the corresponding homomorphism of $F$ into $A_r$, then $\chi_r(\varphi_j(a)) = o_r$ for $j \in J$, $j \neq r$, $a \in A_j$ and $\chi_r(\varphi_r(a)) = a$ for $a \in A_r$. This implies that $\varphi_r$ is one-to-one. Now let $r \in J$, $s \in J$, $r \neq s$, $a \in A_r$, $b \in A_s$, $\varphi_r(a) = \varphi_s(b)$. Then $a = \chi_r(\varphi_r(a)) = \chi_r(\varphi_s(b)) = o_r$ and similarly $b = o_s$. So the images of $\varphi_r$ and $\varphi_s$ have at most one element in common.

Theorem 3 (*on free products*). If the assumptions of theorem 2 are satisfied and if moreover every $A_j (j \in J)$ contains a one-element subalgebra, a $Q$–$V$-algebra $F$ exists, which contains for every $j \in J$ a $V$-subalgebra $A_j'$, $V$-isomorphic to $A_j$, and is generated by the union of the $A_j'$, and with the property that two $A_j'$ with different indices have at most one element in common, and that for every $Q$–$V$-algebra $B$ with a family

$\psi_j (j \in J)$ of $V$-homomorphisms of $A'_j$ into $B$ a $V$-homomorphism $\vartheta$ of $F$ into $B$ exists, which is an extension of $\psi_j$ for all $j \in J$.

An analogous treatment may be given of a generalization of the tensor product of modules or algebra's (cf. e.g. [1]). We start again with a $V$–$W$-axiom system $Q$, which satisfies (A) and with the property that the one-element $V$-algebra is a $Q$–$V$-algebra, and a family $A_j (j \in J)$ of $Q$–$V$-algebras. We denote the direct product of the $A_j$ by $\phi$; the set $\phi$ is the collection of all mappings $f(j)$ with $j \in J$ and $f(j) \in A_j$. We split $V$ into two disjoint sets $V'$ and $V''$ indexed by the index sets $I'$ and $I''$ respectively. The operations of $A_j$ are denoted by $O_{ji} (i \in I'$ or $i \in I'')$ and are $k_i$-ary; the corresponding operation symbols are denoted by $\Omega_i$. For every $f \in \phi$ we take a constant $f'$ and obtain $V_1$ by adding these constants to $V$. The set of these constants can be indexed by $\phi$. For every $i \in I'$, $j \in J$, $b_r \in A_r$, $f_\nu \in \phi$ with $f_\nu(r) = b_r$ $(\nu = 1, \ldots, k_i$, all $r \in J$ with $r \neq j)$ we determine $g \in \phi$ by

$$g(r) = b_r \text{ for } r \neq j,$$
$$g(j) = O_{ji}(f_1(j), \ldots, f_{k_i}(j)).$$

For every $i \in I''$, $f_\nu \in \phi$ $(\nu = 1, \ldots, k_i)$ we determine $g \in \phi$ by

$$g(j) = O_{ji}(f_1(j), \ldots, f_{k_i}(j)).$$

In both cases we put

$$\Omega_i(f'_1, \ldots, f'_{k_i}) = g',$$

and obtain $Q_1$ by adding these identities as axioms to $Q$. Again $Q_1$ satisfies (A).

We call a mapping $\varphi$ of $\phi$ into a $Q$–$V$-algebra $B$ a *polyhomomorphism*, if $\varphi$ is a $V''$-homomorphism of $\phi$ into $B$ and if for every $j \in J$ and $b_r \in A_r$ (all $r \in J$ with $r \neq j$) the mapping $\chi$ of $A_j$ into $B$ defined by $\chi(a) = \varphi(f_a)$ with $f_a \in \phi$ determined by

$$f_a(r) = b_r \text{ for } r \neq j,$$
$$f_a(j) = a,$$

is a $V'$-homomorphism.

Obviously this concept coincides with that of an admissible mapping. The following theorem is a consequence of lemma 1.

Theorem 4 (*on tensor products*). If $Q$ is a $V$–$W$-axiom system, which satisfies (A), if the one-element $V$-algebra is a $Q$–$V$-algebra, if $A_j (j \in J)$ is a family of $Q$–$V$-algebras, and if $\phi$ is the direct product of the $A_j$, a $Q$–$V$-algebra $F$ and a polyhomomorphism $\varphi$ of $\phi$ into $F$ exist, with the property that $F$ is generated by the image of $\varphi$, and that for every $Q$–$V$-algebra $B$ with a polyhomomorphism $\psi$ of $\phi$ into $B$ a $V$-homomorphism $\vartheta$ of $F$ into $B$ exists, satisfying $\vartheta\varphi = \psi$.

The algebra $F$ of theorem 4 is called the *tensor product* of the family $A_j$. If $Q$ is the axiom system of an $R$-module with fixed ring $R$ (every element of $R$ gives an operation of $V$) we get the usual tensor product if we take $V''$ empty. A polyhomomorphism is in this case a multilinear mapping. For the tensor product of algebras (hypercomplex systems), multiplication is to be taken in $V''$ and addition and all scalar multiplications in $V'$.

If $V'$ is empty, a polyhomomorphism is an ordinary homomorphism of $\phi$. In that case $F$ is isomorphic to $\phi$. To prove this we take $B=\phi$ and $\varphi$ the identical mapping. By theorem 4 a homomorphism $\vartheta$ of $F$ into $\phi$ exists with the property that $\vartheta\varphi$ is the identical mapping of $\phi$. This implies that $\varphi$ is an isomorphism of $\phi$ into $F$; as $F$ is generated by the image of $\varphi$, it coincides with this image and so $\phi$ and $F$ are isomorphic. So this case is trivial.

In both cases of free products and tensor products the concept is independent (up to isomorphism) of the choice of the index set $J$, and is associative, as is easily shown.

Finally we discuss the embedding of a $Q$–$V$-algebra into a $Q'$–$V'$-algebra. Let $V$ and $V'$ be sets of operations with $V \subset V'$ and $W$ a set of variables. Let $Q$ be a $V$–$W$-axiom system and $Q'$ a $V'$–$W$–axiom system and $Q \subset Q'$, and let $Q'$ (and therefore $Q$) satisfy (A). Let $A$ be a $Q$–$V$-algebra. We index $V$ with an index set $I$. We denote the operations of $A$ by $O_i$ and the corresponding operation symbols by $\Omega_i (i \in I)$; these operations are $k_i$-ary. For every $a \in A$ we take a constant $a'$ and obtain $V_1$ by adding these constants to $V'$. The set of these constants may be indexed by $A$. For every $a \in A$, $b \in A$ with $a \neq b$ we put

(1) $$a' \neq b',$$

and for every $i \in I$ and $a_\nu \in A\,(\nu = 1, \ldots, k_i)$ we put

(2) $$\Omega_i\,(a_1', \ldots, a_k') = \big(O_i\,(a_1, \ldots, a_{k_i})\big)'.$$

We obtain $Q_1$ by adding all the axioms (1) and (2) to $Q'$.

A mapping of $A$ into a $Q'$–$V'$-algebra is admissible if and only if it is an isomorphism. So every $Q_1$–$V_1$-algebra is $V'$-isomorphic with a $Q'$–$V'$-algebra which contains $A$ as a $V$-subalgebra.

$Q_1$ satisfies (A), if and only if it is consistent. $Q_1$ is consistent, if and only if a $Q'$–$V'$-algebra exists which contains $A$ as a $V$-subalgebra. We get the following theorem.

Theorem 5 (*on embedding*). If $Q$ is a $V$–$W$-axiom system and $Q'$ a $V'$–$W'$-axiom system with $V \subset V'$ and $Q \subset Q'$, if $Q'$ satisfies (A), if $A$ is a $Q$–$V$-algebra and if a $Q'$–$V'$-algebra exists, which contains $A$ as a $V$-subalgebra, a $V'$–$W'$-algebra $F$ exists, which contains $A$ as a $V$-subalgebra and, as a $V'$-algebra, is generated by $A$, and with the property that for every $V'$–$W'$-algebra $B$, which contains $A$ as a $V$-subalgebra, a $V'$-homomorphism $\vartheta$ of $F$ into $B$ exists with the property that the restriction of $\vartheta$ to $A$ is the identical mapping.

In the proof of theorem 5, $Q$ is not used. So we can omit $Q$, and assume

$A$ to be a $V$-algebra. We have added $Q$, as in the applications we are not interested in the embedding of one particular algebra, but of all algebras of a particular algebraic structure.

The only interesting case is that $V' \neq V$, for it $V' = V$, $A$ is a subalgebra of a $Q'$–$V'$-algebra if and only if it is a $Q'$–$V'$-algebra itself. If this is the case, $F = A$ and theorem 5 is trivial.

A drawback of theorem 5 is that one has to know that embedding is possible to be able to apply the theorem. If this requirement is satisfied, theorem 5 yields what one could call a "universal" embedding which may be mapped homomorphically into every other embedding in such a way that all elements of $A$ are invariant. In practice one usually proves the possibility of embedding by constructing the universal embedding.

In some examples this mapping is even an isomorphism. So the embedding of a commutative semigroup with cancellation into a group is always possible and the universal embedding may be mapped isomorphically into every embedding. Theorem 5 gives no information about this fact.

We now discuss the special case that $V$ and $Q$ are empty. Then $A$ is a set without algebraic structure. The conditions imposed on $Q'$ are that it satisfies (A) and that a $Q'$–$V'$-algebra exists with cardinal number not less than the cardinal number of $A$. Theorem 5 yields the existence of a $Q'$–$V'$-algebra $F$, which contains $A$ as a subset and is generated by $A$, with the property that every one-to-one mapping of $A$ into a $Q'$–$V'$-algebra $B$ may be extended to a homomorphism of $F$ into $B$. This is not a new result: the free $Q'$–$V'$-algebra with a number of generators equal to the cardinal number of $A$ satisfies the requirements and by theorem 3 of [2] this free algebra exists under weaker assumptions. If suffices to assume that $Q'$ satisfies (A), and, if the cardinal number of $A$ is $\geqslant 2$, $Q'$ is strongly consistent. Moreover the mapping of $A$ into $B$ need not be one-to-one.

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